Self-interacting scalar field cosmologies: unified exact solutions and symmetries

T. Charters[†], J. P. Mimoso[§]

[†] *Departamento de Engenharia Mecânica/Área Científica de Matemática*

Instituto Superior de Engenharia de Lisboa

Rua Conselheiro Emídio Navarro, 1, P-1949-014 Lisbon, Portugal

Centro de Astronomia e Astrofísica da Universidade de Lisboa

Avenida Professor Gama Pinto 2, P-1649-003 Lisbon, Portugal

[§] [‡] *Departamento de Física, Faculdade de Ciências da Universidade de Lisboa*

Centro de Astronomia e Astrofísica da Universidade de Lisboa

Avenida Professor Gama Pinto 2, P-1649-003 Lisbon, Portugal and

[†]*tca@cii.fc.ul.pt,* [§]*jpmimoso@cii.fc.ul.pt,*

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We investigate a mechanism that generates exact solutions of scalar field cosmologies in a unified way. The procedure investigated here permits to recover almost all known solutions, and allows one to derive new solutions as well. In particular, we derive and discuss one novel solution defined in terms of the Lambert function. The solutions are organised in a classification which depends on the choice of a generating function which we have denoted by $x(\phi)$ that reflects the underlying thermodynamics of the model. We also analyse and discuss the existence of form-invariance dualities between solutions. A general way of defining the latter in an appropriate fashion for scalar fields is put forward.

I. INTRODUCTION

During approximately the past three decades there has been considerable interest in finding exact solutions to the well know scalar field equations in a flat 4-dimensional Friedman-Robertson-Walker (FRW) space-time. This was triggered by the inflationary paradigm [1–4] where a scalar field, dubbed the inflaton, plays a central role in producing a brief stage of accelerated expansion. The search for exact solutions was to a great extent driven by the need to find a scalar field potential which would successfully convey the inflationary prescription, i.e., which would produce sufficient inflation, ending with a graceful exit. Another goal was the establishment of a simple and concise way of relating the dynamics of the scalar field with the CMB observational data and, in particular, to test of the consistency of the slow-roll approximation in the study of generation of fluctuations in the large ϕ field region [5, 6].

The search for exact solutions of models with a self-interacting scalar field has followed diverse

strategies. At first the focal point was the obtainment of approximate solutions for the cases of potentials assumed to be realistic, because of their particle physics motivation. Examples of potentials falling into this class were the spontaneous symmetry breaking double-well potential, the $\lambda\phi^4$ potential, the Coleman-Weinberg potential [1–4]. Another perspective in the search for exact solutions emerged in 1985 by means of which one looked for potentials whose solutions had prescribed properties. Lucchin and Matarrese [7] showed the relation between power-law inflation and the exponential potential, which was subsequently studied in further detail by Halliwell [8], Burd and Barrow [9], Barrow[10], Ratra and Peebles [11]. Other works along this line explored some particular equations of state [12–14] or arbitrary time dependences of the scale factor [15].

A third approach has been to explore several recipes for the formal construction of exact solutions. Muslimov [16], Salopek and Bond [17], and J. Lidsey [18] devised a method to obtain exact solutions using the ϕ scalar field as the independent variable. De Ritis et al [19, 20] explored an alternative method based on the Lie symmetries of the equations to derive some exact solutions.

A skilful combination of the previous methods has lead various authors to derive some classes of new solutions [21–26].

The quest for exact solutions of single field models proceeds at present, indeed, recently scalar field cosmologies have been considered as an explanation of dark energy called quintessence, as well as a way to produce the exotic states dubbed phantom matter [27–30]. In the latter case one envisages the possibility that the kinetic energy of the scalar field be negative. The interest in scalar field exact solutions also extends, naturally, to higher dimensional models [31] and to models based on modified gravity theories such as, Brans-Dicke or superstring and others [32–34].

However two questions have not been fully answered: What is the basic property that allows the equations to be integrated? What are the similarities between all the known solutions?

Here we will be concerned with the question of unifying all the previous solutions in a single, and simple framework. The procedure that we have devised, and which is based on a novel generating function, unifies all the solutions under a single criterion. Furthermore our method, which goes one step beyond earlier attempts to use the ϕ field as an independent variable [16–18], allows us to derive new solutions (which we illustrate in subsection III B), and most importantly provides a way to classify all the solutions, and their corresponding behaviours. Indeed not only does it enable one to make a complete qualitative analysis of the possible asymptotic behaviour that can be expected from single field, scalar field cosmologies, but it also permits us to analyse the form invariance dualities that connect diverse solutions and potentials.

In this paper we work in the framework of General Relativity (GR) and shall consider FRW

models with general N spatial dimensions, although many of the applications some of the applications will be given for $N = 3$ to make connection with the literature. First we show how the equations of motion for a scalar field ϕ in a flat $(N+1)$-dimensional FRW model can be integrated by quadrature. This integration of the equations of motion is not based on any particular technique, but is in fact the result of the simple and particular form of the equations of motion of a scalar field in FRW space-time. The equations of motion of a self-interacting scalar field contain a very special non-linear dissipative term proportional to the square root of the energy of the associated mechanical system. This particular feature is fundamental and underlies the derivation of a large of number of exact solutions, and the construction of particular methods to obtain each of them that can be found in the literature. In the present work we shall present an unified scheme that generates almost every exact solutions for the problem. We illustrate how known-solutions are recovered, and, as an example, we shall obtain a solution which to the best of our knowledge is novel.

In the light of the method that we present here, we also analyse the relations that exist between apparently disconnected sets of solutions. In this regard we extend Chimento's [35] and Chimento and Lazkoz's [36] results, and we discuss a form-invariance symmetry that maps a solution of a given set of equations with a given scalar field potential to another solution of different set of equations associated with some other scalar field potential. This completes the unification of the set of exact solutions of the flat Friedman models with a single scalar field.

II. EQUATIONS OF MOTION

Let us consider a $(N+1)$-dimensional homogeneous and spatially flat spacetime

$$ds^2 = -dt^2 + a^2(t)\sum_{i=1}^{N}(dx^i)^2, \tag{1}$$

where $a(t)$ is the scale factor. Assume that the matter content is a scalar field ϕ which is minimally coupled. The Einstein equations can be written as

$$\ddot{\phi} + NH\dot{\phi} + V_{,\phi} = 0, \tag{2}$$

$$\frac{1}{2}N(N-1)H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{3}$$

where $H = \dot{a}/a$, with an overdot representing the derivative with respect to time. Combining these two equations one obtain

$$\dot{H} = -\frac{\dot{\phi}^2}{N-1}. \tag{4}$$

Introducing the new time variable $d\tau = Hdt$ and the variable

$$x = \frac{\dot{\phi}}{H}, \tag{5}$$

we obtain the following planar, autonomous dynamical system

$$x' = -\left(\frac{1}{2}N(N-1) - \frac{1}{2}x^2\right)\left(\frac{2}{N-1}x + \frac{V_{,\phi}}{V}\right), \tag{6}$$

$$\phi' = x, \tag{7}$$

where the prime stand for the derivative with respect to τ, and $V_{,\phi}$ is the derivative of $V(\phi)$ with respect to ϕ. We study the solutions of (6) and (7) by considering $dx/d\phi$. As

$$\frac{dx}{d\phi} = \frac{x'}{\phi'} = -\frac{\left(\frac{1}{2}N(N-1) - \frac{1}{2}x^2\right)\left(\frac{2}{N-1}x + \frac{V_{,\phi}}{V}\right)}{x} \tag{8}$$

we can write

$$\frac{V_{,\phi}}{V} = \left[-\frac{2x}{N-1} - \frac{2xx_{,\phi}}{N(N-1) - x^2}\right]. \tag{9}$$

Equation (9) can be approached in two alternative ways. On the one hand, given the potential $V(\phi)$ we can, in principle, solve this equation to obtain $x(\phi)$, and thus we can subsequently use (7) to obtain a solution. But this is only strictly possible for the case of the exponential potential, as discussed below. On the other hand, we can instead arbitrarily choose $x = x(\phi)$, and in the sequel obtain the corresponding scalar field potential, as well as an exact solution through the use of (7). In this latter case, the integration of equation (9) yields

$$V(\phi) = A\left(\frac{1}{2}N(N-1) - \frac{1}{2}x^2(\phi)\right) e^{-\frac{2}{N-1}\int x(\phi)d\phi} \tag{10}$$

where A is an integration constant which determines the amplitude of the potential. The specification of a particular form of $x(\phi)$ (satisfying $x^2 \leq N(N-1)$) thus gives the explicit solution

$$\int d\tau = \int \frac{d\phi}{x(\phi)}, \tag{11}$$

where one uses equation (7), and further utilisation of (3) yields

$$H(\phi) = \pm\sqrt{A} e^{-\frac{1}{N-1}\int x(\phi)d\phi}. \tag{12}$$

Notice that we also have

$$\dot{\phi}^2 = x^2(\phi) e^{-\frac{2}{N-1}\int x(\phi)d\phi}. \tag{13}$$

Returning to time t, one gets,

$$\int dt = \int \frac{d\phi}{\pm\sqrt{A}x(\phi)\exp\left(-\frac{1}{N-1}\int x(\phi)d\phi\right)}, \tag{14}$$

which is the final quadrature.

By selecting a suitable $x = x(\phi)$, and hence $H(\phi)$ from equation (12), one easily integrates the equations for the scalar field model in closed form. We obtain the solution explicitly in t cosmic time, provided that equation (14) is invertible. As we shall illustrate in the following section the many solutions sparsely found in the literature result from specific, simple choices of $x(\phi)$. Our result naturally shows how the set of existing solutions can be extended.

From the definition (5) we see that $x(\phi)$ is related to the usual barotropic index $\gamma = (\rho_\phi + p_\phi)/\rho_\phi$, where

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi), \tag{15}$$

$$p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi), \tag{16}$$

in the following way

$$x(\phi) = \pm\sqrt{\frac{N(N-1)}{2}\gamma(\phi)}, \tag{17}$$

and thus choosing $x(\phi)$ amounts to a specification of the equation of state. However, the present formalism clearly reveals that the choice of the potential $V(\phi)$ does not uniquely determine the equation of state. Indeed, from equation (9), we realise that the specification of a potential is associated with families of solutions $x(\phi)$ which will differ through integration constants, and hence, given (17), will correspond to different thermodynamic regimes.

If, for instance, one considers $x(\phi) = \lambda$ then it follows directly from equation (10) that $V(\phi) \propto \exp(-2\lambda\phi/(N-1))$ (see also Section III), and the exact solution takes the form

$$\begin{aligned} \phi(t) &= \frac{N-1}{\lambda}\ln(t-t_0), & (18)\\ H(t) &= \pm\frac{(N-1)}{\lambda^2}\frac{1}{t-t_0}, & (19)\\ V(\phi) &= A\left(\frac{N(N-1)}{2} - \frac{\lambda^2}{2}\right)e^{-\frac{2\lambda}{N-1}\phi}. & (20) \end{aligned}$$

Notice that the particular case where $\lambda = 0$ reduces to the case of a cosmological constant, $V(\phi) = AN(N-1)/2 \equiv \Lambda$, and we recover the de Sitter solution [14].

But if one assumes instead, from the start, the potential to be exponential (20), then equation (9) is integrable, and we derive the most general $x(\phi)$ generating function that yields the exponential

potential. We obtain though $x(\phi)$ in implicit form

$$(\beta - x)^{-(\beta\alpha+\lambda)}(\beta + x)^{\beta\alpha-\lambda}(\alpha x - \lambda)^{2\alpha\lambda} = \Gamma_0 \exp\left(\beta^2\alpha^2 - \lambda^2\right) \tag{21}$$

where we have defined $\alpha = 2/(N-1)$ and $\beta = \sqrt{N(N-1)}$, and where Γ_0 is an arbitrary integration constant. The latter expression cannot be solved to yield an explicit form for $x(\phi)$.

However, we see that we have only two possible choices regarding the limit behaviours of $x(\phi)$ for the exponential potential for the asymptotic limit $\phi \to \infty$. Consider the change of variables $\chi = 1/\phi$ [37], then the planar system (6) and (7) can be recast in the form

$$x' = -\left(\frac{1}{2}N(N-1) - \frac{1}{2}x^2\right)\left(\frac{2}{N-1}x - \lambda\right), \tag{22}$$

$$\chi' = -\chi^2 x, \tag{23}$$

which shows that the infinity manifold $\chi = 0$, or equivalently $\phi = \infty$, is invariant. When $x = x_1 = \lambda(N-1)/2$ or $x^2 = (x_2)^2 = N(N-1)$, $x' = 0$ also vanishes, and these two solutions correspond to the asymptotic, equilibrium solutions. They give completely different asymptotic regimes, both dynamical and thermodynamical. A local stability analysis reveals that on the infinity manifold, x_1 is stable and x_2 unstable for $\lambda^2 < 4N/(N-1)$ (otherwise, they will be respectively unstable and stable).

When $V(\phi)$ is not exponential we should distinguish the equilibrium solutions of the system into those arising at finite values of ϕ and those associated with the asymptotic limit $\phi \to \infty$. In the former case, equation (7) yields $x = 0$ as a necessary condition for a equilibrium point to occur at finite values of ϕ. Substitution into the equation (6) gives

$$x' = -\frac{N(N-1)}{2}\frac{V_{,\phi}}{V}, \tag{24}$$

and thus the left-hand side vanishes if and only if $V_{,\phi} = 0$, i.e., whenever $V(\phi)$ has an extremum, say $\phi = \phi_0$. A straightforward linear stability analysis reveals that the equilibrium point is stable (unstable) if $V(\phi_0)$ is a minimum (respectively, a maximum).

On the other hand, if the equilibrium point occurs when $\phi \to \infty$, the introduction of $\chi = 1/\phi$, as done before, shows that an asymptotic behaviour associated with a non-vanishing constant value of x only happens if $V(\phi)$ asymptotes to an positive exponential behaviour, i.e., $\frac{2}{N-1}x \to \lambda$ in equation (23). In this latter case the discussion of the stability previously produced for the exponential case applies.

Still regarding $\phi \to \infty$ case, we also have, of course, the possibility that the equilibrium point be characterized by $x = 0$ when $V_{,\phi} \to 0$, and hence, $V \to \Lambda$ corresponding to the de Sitter behaviour.

Looking at equations (6) and (7) it is easy to see that it is only for the exponential potential that these two equations decouple. For an arbitrary potential $V = V(\phi)$ this does not occur, and thus the condition for decoupling is given by the potential in the form (10), for a given $x = x(\phi)$. This leaves us with only one equation to integrate $\phi' = x(\phi)$. It seems then natural to assume that it should exist a transformation that maps the free-field solution, namely $x(\phi) = cte$, to any other solution with a prescribed $x(\phi)$. This is the nature of the invariance-form transformation that we are going to construct.

For completeness let us comment on the slow-roll approximation which is a central (if not a starting) assumption of almost every scenario in inflationary cosmology. From appropriate combinations of the Hubble parameter and of its derivatives one can define quantities which take small values when the slow roll regime of the scalar field dynamics holds. These are called slow-roll parameters, and subsequently the expressions of relevant quantities of the model, such as the spectral scalar and tensor indexes, are written as expansions in terms of these parameters in the neighbourhood of the slow-roll regime [6]. Typically, however, only the first few enter into any expressions of interest.

In terms of the generating function $x(\phi)$ that we have introduced, the first two slow-roll parameters are

$$\epsilon(\phi) = \frac{1}{2}x^2(\phi), \tag{25}$$

$$\eta(\phi) = \frac{1}{2}x^2(\phi) - \frac{dx}{d\phi}(\phi). \tag{26}$$

Apart from a constant of proportionality, ϵ measures the relative contribution of the field's kinetic energy to its total energy. The quantity η, on the other hand, measures the ratio of the field's acceleration relative to the friction term acting on it due to the expansion of the universe. The slow-roll approximation applies when these quantities are small in comparison to unity, that is, when we have both $x^2(\phi) \ll 1$ and $\frac{dx}{d\phi}(\phi) \ll 1$. This reduces the dynamical system (6) and (7) to

$$x' \simeq -\frac{1}{2}N(N-1)\left(\frac{2}{N-1}x + \frac{V_{,\phi}}{V}\right), \tag{27}$$

$$\phi' = x, \tag{28}$$

and this gives

$$\frac{dx}{d\phi} \simeq -N - \frac{V_{,\phi}}{xV} \tag{29}$$

and so $\eta \ll 1$ implies that $\frac{V_{,\phi}}{xV} \simeq -N$, i.e., locally $V(\phi) \sim \exp\left[-N\int x(\phi)d\phi\right] \sim 1 - N\int x(\phi)d\phi$.

III. EXACT SOLUTIONS

In what follows we list some of the most important exact solutions that can be found in the literature, and show how they arise as particular cases of the previous scheme by giving the choice of $x(\phi)$, which in turn determines the form of the potential $V(\phi)$ for each case. This illustrates how the present method recovers, and unifies the many solutions that populate the literature. These solutions rely on the fact that (14) can be inverted, and this together with the fact that in all cases $x(\phi)$ can be cast as a logarithmic derivative, $x(\phi) = g'(\phi)/g(\phi)$ where $g(\phi)$ is some well-behaved function of ϕ, are a common feature for all of them. (Since in this section our goal is to establish the connection between our procedure and the literature on exact solutions, in what follows we restrict to the $N = 3$ case, and we do not review the details of the solutions. For the latter details the reader is kindly referred to the quoted references).

In the second subsection below, we also derive a new exact solution, and discuss its main properties. This enables us to illustrate how the generating method introduced in the present work permits to expand the set of known solutions, and is not limited to reproduce them.

A. List of some of exact solutions

1. Direct solutions

1. One of the most important exact solutions is the well known power-law solution which has been derived in many forms and through different methods [7]. As we have seen in the previous section, this solution corresponds to the choice $x(\phi) = \lambda$, where λ is a constant, and yields (for the $N = 3$ case)

$$V(\phi) = A\left(3 - \lambda^2/2\right) e^{-\lambda\phi}, \tag{30}$$

   where A is an arbitrary constant that fixes the height of the potential (in what follows we will adopt this notation for the constant factor that defines the amplitudes of the various potentials). Note that this solution also permits to integrate the scalar perturbations giving a constant scalar spectral index [5, 6].

2. The Easther solution [23] corresponds to the choice $x(\phi) = -\phi$ and yields the potential

$$V(\phi) = A\left(3 - \phi^2/2\right) e^{\phi^2/2}, \tag{31}$$

where, as explained, A is an arbitrary constant, and subsequently gives

$$a(\phi) = \frac{\phi_0}{\phi}, \tag{32}$$

$$t(\phi) = \frac{1}{2\sqrt{A}}\left[\mathrm{Ei}\left(-\frac{\phi_0^2}{4}\right) - \mathrm{Ei}\left(-\frac{\phi^2}{4}\right)\right], \tag{33}$$

where Ei is the exponential integral function [38]. This solution has the remarkable feature that it also yields constant scalar spectral index which is equal to 3.

The Easther solution is a particular case of the class of solutions characterized by $x = \lambda\phi$. In the latter case the solutions are characterized by

$$V(\phi) = A\left(3 - \lambda^2\phi^2/2\right)e^{\lambda\phi^2/2}, \tag{34}$$

and

$$a(\phi) = a_0\phi^{1/\lambda}, \tag{35}$$

$$t(\phi) = \frac{1}{2\lambda\sqrt{A}}\left[\mathrm{Ei}\left(\frac{\lambda\phi_0^2}{4}\right) - \mathrm{Ei}\left(\frac{\lambda\phi^2}{4}\right)\right], \tag{36}$$

where Ei is, once again, the exponential integral function.

3. The intermediate inflationary solution [12] is given by $x(\phi) = \beta/\phi$, where β is a constant, and is one of the most famous solutions. The scalar field potential is given in this case by

$$V(\phi) = \frac{16A^2}{(\beta+4)^2}\left(3 - \frac{\beta^2}{2\phi^2}\right)\left[\frac{\phi}{(2A\beta)^{1/2}}\right]^{-\beta}, \tag{37}$$

and we have

$$a(t) = \exp\left(At^f\right) \tag{38}$$

$$\phi = \left(2A\beta t^f\right)^{1/2}, \tag{39}$$

where f is a constant such that $0 < f < 1$, and $\beta = 4(f^{-1} - 1)$.

4. In [22] one finds the potentials

$$V_1(\phi) = A^2\lambda^2\left[(3A^2 - 2)\cosh^2(\phi/A) + 2\right] \tag{40}$$

and

$$V_2(\phi) = \frac{1}{12}\lambda^2 A^{-2}\phi^2(\phi^2 + A^2)(\phi^4 A^{-4} + A - 6 + 2\phi^2) \tag{41}$$

which correspond, respectively, to

$$a_1(t) \;=\; a_0\,[\sinh(2\lambda t)]^{A^2/2} \tag{42}$$

$$\phi_1(t) \;=\; A\ln[\tanh(\lambda t)] \tag{43}$$

$$x_1(\phi) \;=\; -2A^{-1}\tanh(\phi/A), \tag{44}$$

and to

$$a_2(t) \;=\; a_0\,[\sinh(2\lambda t)]^{A^2/2}\exp[-A^2\coth^2(\lambda t)/12], \tag{45}$$

$$\phi_2(t) \;=\; A\mathrm{csch}(\lambda t), \tag{46}$$

$$x_2(\phi) \;=\; -\frac{6\phi}{A^2+\phi^2}. \tag{47}$$

In the previous epxressions A and λ are arbitrary constants.

5. In [24] one finds a class of solutions with

$$\phi(t) \;=\; A\exp(-\mu t^n), \tag{48}$$

parametrized by n constant, to which we associate the choice

$$x(\phi) = -2\frac{\phi\,(\log(\phi/A))^{1-1/n}}{\int\phi\,(\log(\phi/A))^{1-1/n}\,d\phi}. \tag{49}$$

In Ref. [21] another family of solutions, also parameterized by constant n, is displayed such that

$$\phi(t) = A(\ln t - B)^n. \tag{50}$$

this family of solutions is recovered with the choice

$$x(\phi) = -2\frac{(\phi/A)^{1/n}\exp((\phi/A)^{1/n})}{\int(\phi/A)^{1/n}\exp((\phi/A)^{1/n})d\phi}. \tag{51}$$

In both cases the general form of the potentials is rather involved, except for some simple cases arising form the restriction of n to take some particular values, and thus we refer the reader to the cited references for further details about the solutions.

2. *Pair defined solutions*

There are other solutions for which a special method was constructed. The following two examples provide a closer look at the procedures that were used.

1. In [39] the authors make an assumption which amounts in our prescription to the choice

$$x(\phi) = \frac{[1 - F^2(\phi)]\beta^2(\beta - 1)^2}{4 - [1 - F^2(\phi)]\beta(\beta - 1)} \tag{52}$$

where β is a constant, and thus derived the potential

$$V(\phi) = \exp\left(\mp 2\beta \int \sqrt{\frac{F(\phi) - 1}{F(\phi) + 1}} d\phi\right). \tag{53}$$

The method provides an exact solution with the further prescription of

$$F(\phi) = \cosh(\lambda\phi), \tag{54}$$

where λ is a constant, then yielding

$$V(\phi) = A(1 + \cosh(\lambda\phi))^{\mp\beta/\lambda - 1}. \tag{55}$$

Putting $\beta/\lambda = 2$ and choosing the positive sign in (53) one gets the solution

$$\phi(t) = \frac{1}{\lambda} \ln\left[\frac{\exp(\lambda\sqrt{A}t) + 1}{\exp(\lambda\sqrt{A}t) - 1}\right]. \tag{56}$$

2. Reference [40] provides another special method which is equivalent to choosing

$$x(\phi) = \pm\sqrt{\frac{2g(H)}{H^2}} \tag{57}$$

for which one has

$$V(\phi) = 3H^2(\phi) - g(H(\phi)). \tag{58}$$

The solutions of [40] were obtained with the choices

$$g_1(H) = -AH^n, \tag{59}$$
$$g_2(H) = \pm\frac{4}{C}\sqrt{ACH - H^2}H. \tag{60}$$

Notice that the authors also derived the solution given in [22] by using a specific choice of the function $g(H)$.

3. In [26] a special method was given for a $(N + 1)$-dimensional FRW space-time using certain classes of generating functions using

$$x(\phi) = (N - 1)\frac{G(\phi)}{\int G(\phi)}, \tag{61}$$

where $G(\phi)$ is a function so that $G(\phi) = \alpha H(\phi) + L(\phi)$, where $H(\phi)$ and $L(\phi)$ are functions specified below, and α is a constant.

(a) For H-linear generating functions and for any N, we recover the solutions of [26] with the following choice of $x(\phi)$

i.

$$x(\phi) = -(N-1)\frac{L(\phi)e^{\alpha\phi/(N-1)}}{\displaystyle\int L(\phi)e^{\alpha\phi/(N-1)}} + \alpha \tag{62}$$

where

$$L(\phi) = D\alpha e^{-\alpha\phi/(N-1)} \sum_{j=m}^{m} \frac{j}{j+1} e^{-j\alpha\phi/(N-1)}, \tag{63}$$

for the cases $m = n = 1$ and $m = n = 2$.

ii. And also $L(\phi) = c_1\phi + c_2\phi^2$ when $c_1, c_2 \in \mathbb{R}$.

(b) For the method of the multiplicative generating functions of Ref. [26], the choices of $x(\phi)$ become

i. $G(\phi) = \omega\phi^n$ and thus

$$x(\phi) = \frac{\omega\phi^n}{cte + \omega\phi^{n+1}/(n+1)}. \tag{64}$$

ii. $G(\phi) = \omega\phi^n \exp(-\lambda\phi^m)$ and thus

$$x(\phi) = \frac{e^{-\lambda\phi^m} m(N-1)\left(\lambda\phi^m\right)^{\frac{n+1}{m}}}{\phi\Gamma\left(\frac{n+1}{m}, \lambda\phi^m\right)} \tag{65}$$

The special cases were considered: $n = -3$, $m = -2$ and $\lambda = 1/2$ for $H_0 = 0$ and $\omega = -2/\sqrt{3}$; $H_0 = 1/2$ and $\omega = 1 + 2/\sqrt{3}$ and with $N = 3$.

iii. $G(\phi) = \omega\phi^n(1 - \lambda\phi^m)^\mu$ and thus

$$x(\phi) = \frac{(n+1)(N-1)\left(1-\lambda\phi^m\right)^\mu}{\phi\ {}_2F_1\left(\frac{n+1}{m}, -\mu; \frac{m+n+1}{m}; \lambda\phi^m\right)} \tag{66}$$

The special cases were considered: $n = 1$, $m = 2$ and $\lambda = \mu = \omega = 1/2$ with $H_0 = 1/(24\lambda)$ and with $N = 3$.

4. There are also exact solutions for string motivated models in [41]. The corresponding choices are

$$x_1(\phi) = 2\xi\frac{A - 2Be^{-\xi\phi}}{A - Be^{-\xi\phi}}, \tag{67}$$

$$x_2(\phi) = -2\xi\frac{A + 2Be^{-\xi\phi}}{A + Be^{-\xi\phi}}, \tag{68}$$

$$x_3(\phi) = 2\xi\frac{A - 3Be^{-2\xi\phi}}{A - Be^{-2\xi\phi}}, \tag{69}$$

where ξ, A and B are arbitrary constants, which yield the following potentials

$$V_1(\phi) = A^2(3-2\xi^2)e^{-2\xi\phi} + 2AB(4\xi^2-3)e^{-3\xi\phi} + B^2(3-8\xi^2)e^{-4\xi\phi}, \tag{70}$$

$$V_2(\phi) = A^2(3-2\xi^2)e^{-2\xi\phi} - 2AB(4\xi^2-3)e^{-3\xi\phi} + B^2(3-8\xi^2)e^{-4\xi\phi}, \tag{71}$$

$$V_3(\phi) = A^2(3-2\xi^2)e^{-2\xi\phi} + 6AB(2\xi^2-1)e^{-4\xi\phi} + 3B^2(1-6\xi^2)e^{-6\xi\phi}, \tag{72}$$

respectively, and the associated solutions given by:

$$t_1(\phi) = \frac{1}{A\xi^2}\left[\frac{1}{2}\left(e^{-\xi\phi} - e^{-\xi\phi_0}\right) + \frac{B}{A}\ln\left(e^{-\xi(\phi-\phi_0)}\frac{A-2Be^{-\xi\phi}}{A-2Be^{-\xi\phi_0}}\right)\right], \tag{73}$$

$$t_2(\phi) = \frac{1}{A\xi^2}\left[\frac{1}{2}\left(e^{-\xi\phi} - e^{-\xi\phi_0}\right) - \frac{B}{A}\ln\left(e^{-\xi(\phi-\phi_0)}\frac{A+2Be^{-\xi\phi}}{A+2Be^{-\xi\phi_0}}\right)\right], \tag{74}$$

$$t_3(\phi) = \frac{1}{2A\xi^2}\left[\left(e^{-\xi\phi} - e^{-\xi\phi_0}\right) + \frac{C}{2}\ln\left(\frac{(1-Ce^{-\xi\phi})(1+Ce^{-\xi\phi_0})}{(1-Ce^{-\xi\phi_0})(1+Ce^{-\xi\phi})}\right)\right], \tag{75}$$

where $C = \sqrt{3B/A}$.

B. A new exact solution

We now derive a new exact solution by exploring the functional forms of (12) and (10) uncovered by the properties of the Lambert function [42]. Since the solution is novel we revert to the N-dimensional case, and later, when appropriate, we shall restrict it to the $N = 3$ case.

The Lambert function is defined to be the function satisfying

$$W(\phi)e^{W(\phi)} = \phi, \tag{76}$$

and it is used in many applications [42]. When ϕ is real, for $-1/e \leq \phi < 0$ there are two possible real values of $W(\phi)$ [42]. We denote just by $W(\phi)$ the branch satisfying $-1 \leq W(\phi)$. Differentiating the defining equation (76), and solving for W', we obtain the following expression for the derivative of W:

$$W'(\phi) = \frac{1}{(1+W(\phi))\exp(W(\phi))} \tag{77}$$

$$= \frac{W(\phi)}{\phi(1+W(\phi))}, \quad \phi \neq 0. \tag{78}$$

(79)

Another useful relation is the integral of $W(\phi)$ given by

$$\int W(\phi)d\phi = \phi\left(W(\phi) + \frac{1}{W(\phi)} - 1\right). \tag{80}$$

If $x(\phi)$ is chosen such that

$$\int x(\phi)d\phi = -(N-1)W(f(\phi)), \tag{81}$$

then, according to equations (77) and (81), the pair $(V(\phi), H(\phi))$ that solves the equations of motion is given by

$$V(\phi) = A\left[\frac{N(N-1)}{2}\frac{f^2(\phi)}{W^2(f(\phi))} - \frac{(N-1)^2}{2}\frac{f'^2(\phi)}{(1+W(f(\phi)))^2}\right], \tag{82}$$

$$H(\phi) = \pm\sqrt{A}e^{W(f(\phi))}, \tag{83}$$

from which we derive a first integral

$$\left(\frac{1+W(f(\phi))}{f'(\phi)}\right)^2 \dot{\phi}^2 = A(N-1)^2. \tag{84}$$

Another first integral is the Friedman equation (3).

In what follows we consider the $N = 3$ case. In order to find a solution we take $f(\phi) = \phi$ in (81) and in the following equations. Thus we have $x(\phi) = -2W'(\phi)$ where $W(\phi)$ is the Lambert function and where A is a constant, In this case one gets for the self-interacting potential $V(\phi)$ the expression

$$V(\phi) = A\left(3 - \frac{1}{2}\frac{W(\phi)^2}{\phi^2(1+W(\phi))^2}\right)\frac{\phi^2}{W^2(\phi)}. \tag{85}$$

In Figure 1 we show a plot of this potential as a function of ϕ.

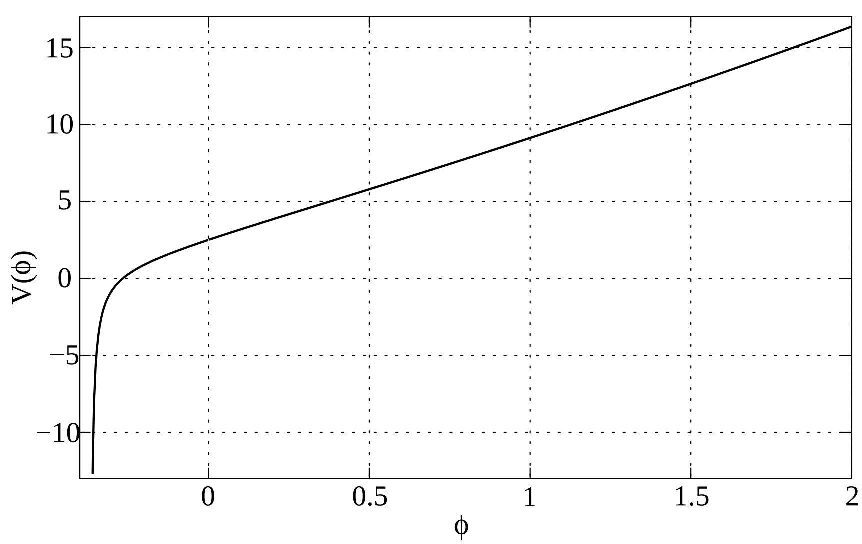

FIG. 1: The self-interacting potential as a function of $\phi > -1/e$.

The Hubble parameter reads

$$H(\phi) = \pm\sqrt{A}e^{W(\phi)}, \tag{86}$$

and we have plotted H as a function of ϕ considering the expanding branch in (86) (the plus sign).

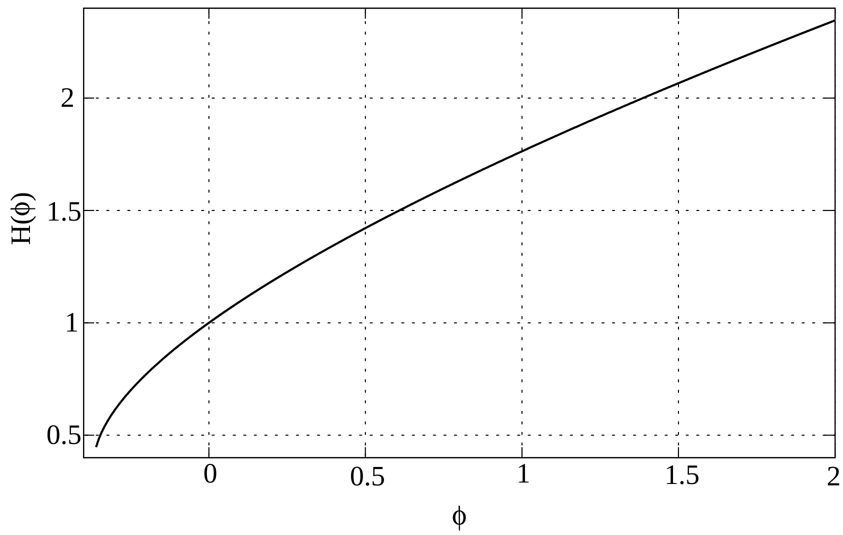

FIG. 2: The Hubble parameter as a function of $\phi > -1/e$.

Using the definition $x = \dot{\phi}/H$, it is easy to see that the velocity of the ϕ field is proportional to the condition number of the Lambert function, that is,

$$\dot{\phi} = \pm 2\sqrt{A}\frac{\phi W'(\phi)}{W(\phi)} = \pm 2\sqrt{A}\frac{1}{1+W(\phi)}, \tag{87}$$

and using (80), one gets,

$$\pm\, 2\sqrt{A}(t-t_0) = \phi\left(\frac{1}{W(\phi)} + W(\phi)\right). \tag{88}$$

These equations cannot be easily inverted, but two different asymptotic solutions can be obtained. If $\phi \ll e$ then (88) yields

$$\phi(t) = \pm 2\sqrt{A}(t-t_0)\ln\left(\pm 2\sqrt{A}(t-t_0)\right). \tag{89}$$

For $\phi \gg e$ one gets

$$\phi(t) = 2\sqrt{A}\frac{t-t_0}{2W(\pm\sqrt{2}/2\sqrt[4]{A}(t-t_0))}. \tag{90}$$

Because

$$W(\phi) \sim \log\phi - \log(\log(\phi)), \quad \phi \gg 1 \tag{91}$$

the asymptotic form of the potential for large values of ϕ is given by

$$V(\phi) \sim A\left[\frac{3\phi^2}{\log(\phi/\log\phi)} - \frac{1}{2\left(1+\log(\phi/\log\phi)\right)^2}\right], \tag{92}$$

which is thus the form of the potential V for the slow-roll regime.

The slow-roll parameters [5, 6, 43] read

$$\epsilon = 2\frac{W^2(\phi)}{\phi^2(1+W(\phi))^2}, \tag{93}$$

$$\eta = 2\frac{W^2(\phi)}{\phi^2(1+W(\phi))^2} - 2\frac{e^{-2W(\phi)}(2+W(\phi))}{(1+W(\phi))^3}. \tag{94}$$

Notice that $\epsilon \leq 2$ for $\phi > 0$ and that $\epsilon \to 0$ as $\phi \to +\infty$. In Figure 3 we show the values of the first and second slow-roll parameters as a function of ϕ, notice that both converge to zero when ϕ goes to infinity. So we see that slow-roll inflation takes place at large values of ϕ. At $\phi = 0$, i.e., for small values of ϕ, the model has a radiation-like behaviour (since $\epsilon = 3\gamma/2$, and hence $\gamma = 4/3$ implies $\epsilon = 2$).

We recall that the condition for inflation to occur is $\epsilon < 1$ and that it ends at $\epsilon = 1$. If we assume that the scalar field has a large initial value the inflationary period ends for the value of ϕ^* that solves the non-linear equation, see (93),

$$W'(\phi) = \frac{\sqrt{2}}{2}. \tag{95}$$

It turns out that in this case the value of ϕ that corresponds to the end of inflation can be determined explicitly. This is one of the most remarkable properties of the Lambert function. In order to obtain ϕ^* consider equation (78). Invert, and then multiply both sides by e and notice that

$$1 + W(\phi) = W(e\sqrt{2}), \tag{96}$$

using $\phi = W(\phi)e^{W(\phi)}$ one gets

$$\phi^* = \sqrt{2}\left(1 - \frac{1}{W(e\sqrt{2})}\right) \simeq 0.21631 \tag{97}$$

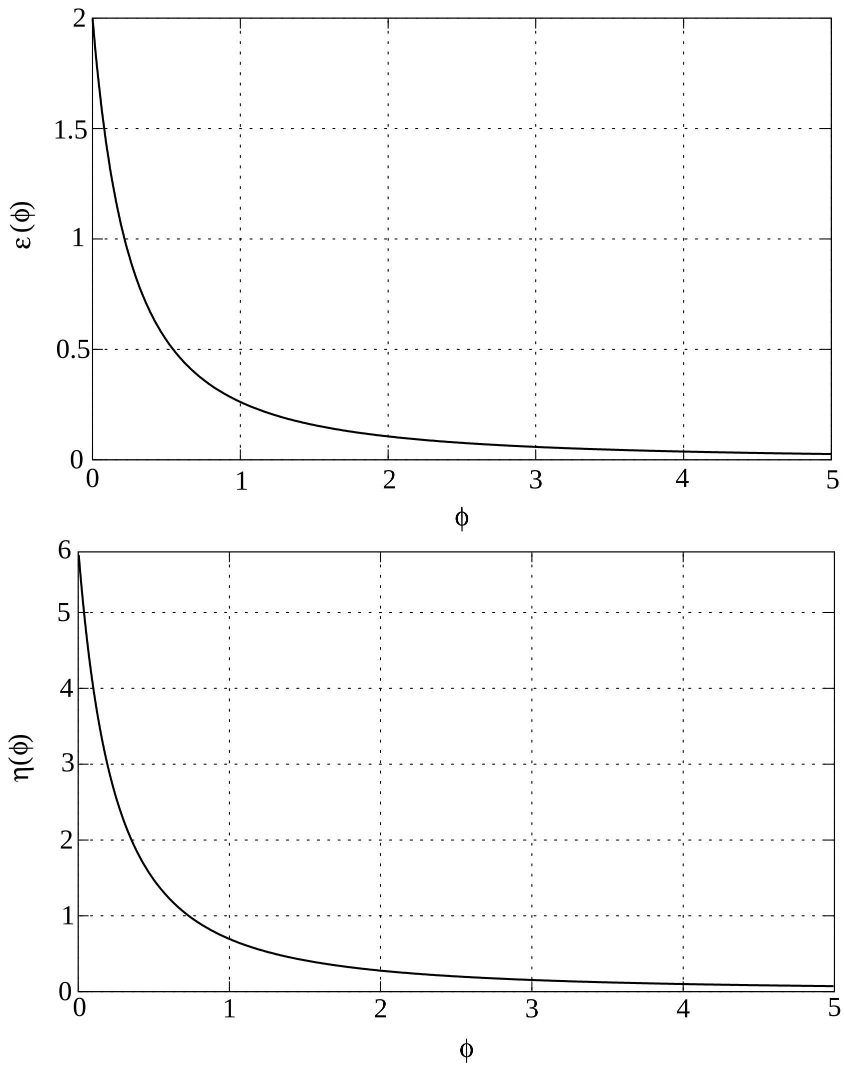

FIG. 3: The first two slow-roll parameters as a function of ϕ.

Let us now study the behaviour of the spectral and tensor indexes in terms of the first order slow-roll expansion [5, 6, 43]. This first order slow-roll expansion is sufficient for our present needs. These values of the spectral and tensor indexes are

$$n_S \; = \; 1 - 4(W'(\phi))^2 + 4W''(\phi) \tag{98}$$

$$n_T \; = \; -4(W'(\phi))^2. \tag{99}$$

In Figure 4 we show the plots for both indexes. The smallest value of the scalar field is ϕ^* in these two plots.

Notice that for large values of ϕ we get $n_S \simeq 1$ and $n_T \simeq 0$. Therefore this novel scalar field model

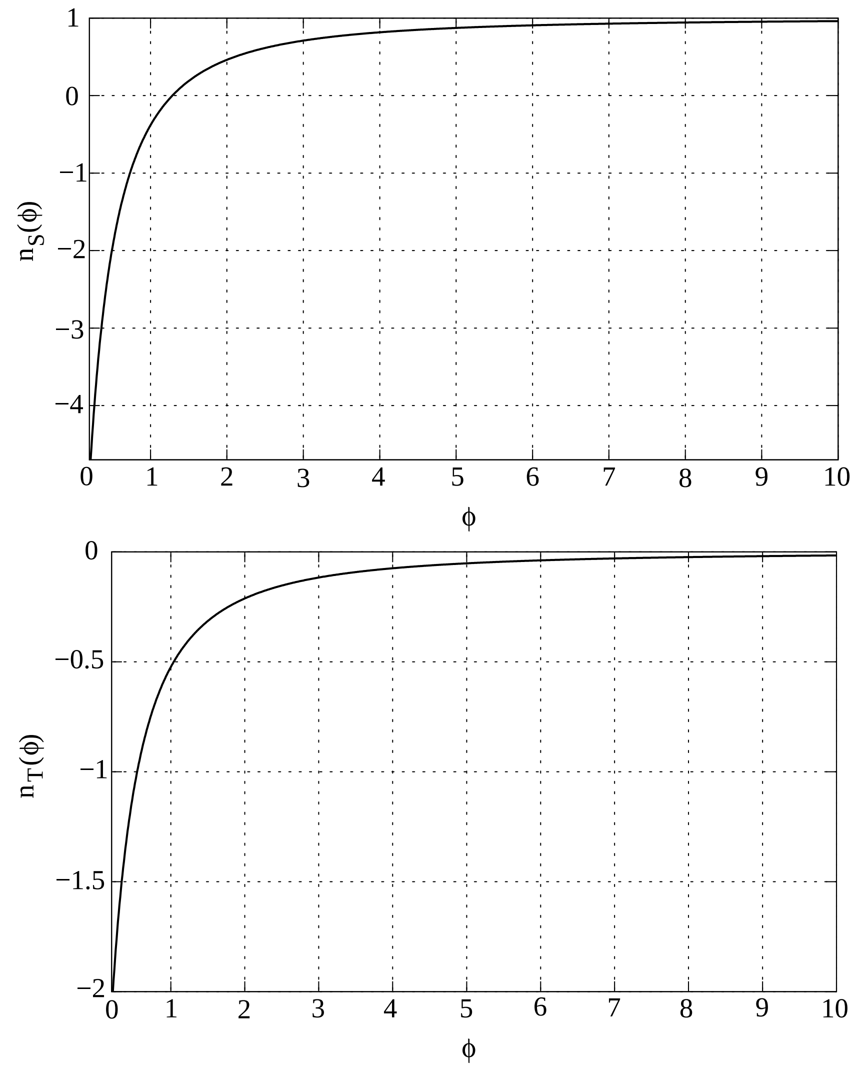

FIG. 4: The spectral and tensor indexes as a function of ϕ until the end of the inflationary regime.

based on the choice of $x(\phi)$ given by equation (81), provides another example of a model that yields a perfectly scale-invariant Harrison-Zeldovich spectrum for large field inflation. Interestingly also, the rolling down of the potential brings the model towards a radiation-like behaviour for small values of ϕ.

IV. FORM-INVARIANCE MAP

In this section we address the question: What are the similarities between all the known solutions? In other words, given that we have shown that the exact solutions are associated with different choices of the generating function $x(\phi)$, is it possible to go from one solution to another solution by means of a transformation $x(\phi) \to \bar{x}(\bar{\phi})$? The answer to this question is related to

the classical technique of obtaining solutions of ordinary differential equations involving a comparison equation and the construction of a map that performs the transformation from a comparison equation to another final equation [44].

The underlying question is whether there is a form-invariance transformation between different solutions, that is an invertible map, which we denote by Ψ, that preserves the form of the equations of motion. In recent articles, [45] and [35], this type of symmetry has been discussed. It was shown that the Einstein equations of a flat FRW space-time with a perfect fluid

$$\frac{N(N-1)}{2}H^2 = \rho, \tag{100}$$

$$\dot{\rho} + NH(\rho + p) = 0, \tag{101}$$

where ρ is the energy density, p the pressure, and $H = \dot{a}/a$, admit a non-trivial map, called a form-invariance transformation, which transforms the quantities (a, H, ρ) to $(\bar{a}, \bar{H}, \bar{\rho})$, so that the latter quantities satisfy the equations

$$\frac{N(N-1)}{2}\bar{H}^2 = \bar{\rho}, \tag{102}$$

$$\dot{\bar{\rho}} + N\bar{H}(\bar{\rho} + \bar{p}) = 0. \tag{103}$$

The Ψ map is given by

$$\Psi_\rho : (\rho, H, p) \longrightarrow \left(F(\rho), ((F(\rho)/\rho)^{1/2} H, -(F(\rho) + (\rho/(F(\rho))^{1/2} (\rho + p)dF(\rho)/d\rho\right), \tag{104}$$

where F is an invertible function, that is,

$$\rho \rightarrow \bar{\rho} = F(\rho) \tag{105}$$

$$H \rightarrow \bar{H} = ((F(\rho)/\rho)^{1/2} H \tag{106}$$

$$p \rightarrow \bar{p} = -F(\rho) + (\rho/(F(\rho))^{1/2} (\rho + p)dF(\rho)/d\rho. \tag{107}$$

equation (105) defines the transformation, equation (106) guarantees the form-invariance of the Friedmann equation (100), and finally equation (107) imposes the form-invariance of the conservation equation (101).

If we now consider perfect fluids with a barotropic equation of state $p = (\gamma - 1)\rho$, where γ is the barotropic index, the indexes of both fluids are related by

$$\bar{\gamma} = \left(\frac{\rho}{F(\rho)}\right)^{3/2} \frac{dF(\rho)}{d\rho}\gamma. \tag{108}$$

An application of this form-invariance transformation to the case of a self-interacting scalar field was given in [36] for the case where $\bar{\rho} \propto \rho$. Chimento and Lazkoz have shown that it is possible

to transform standard scalar field cosmologies into phantom cosmologies. This phantom duality transformation is illustrated by the case of the exponential potential, which as it is well known, is associated with $\bar{\rho} \propto \rho$, and constant γ, hence inducing a power-law solution (see equations (18)–(20)). However, no general relation similar to (104) or (108) was given connecting any pair of self-interacting scalars fields yielding other types of solutions.

Here we aim at analysing the general form-invariance dualities that may be established between any scalar field solutions. Our formalism is better suited to this purpose than that of reference [36], since it relies on the quantities that indeed characterise the scalar field solutions as we have shown in the first part of this paper. The quantities considered in (104) stem from a fluid description which does not separate well the roles played by the kinetic and potential energies of the scalar field since they are combined in ρ_ϕ. Establishing a form-invariance duality requires both the assumption of the transform (105) and that of an equation of state for the original model, which then yield the dual model quantities by means of the Eqs .(106) and (107). In the case of a scalar field model, the method of [36] requires that one be able to integrate back from the assumption of an equation of state the corresponding form of the potential, a task which can be cumbersome. In our case the form fo the potential is readily available, since it follows directly from the specification of $x(\phi)$.

We recall that a scalar field can be interpreted as a perfect fluid with the well known correspondence given by equations (15) and (16), and thus it is preferable to use the quantities $(H, x = \dot{\phi}/H, \phi)$ instead of H, ρ and p. Our aim is to explicitly construct the map

$$\Psi_\phi : (H, x, \phi) \longrightarrow (\bar{H}, \bar{x}, \bar{\phi}), \tag{109}$$

such that the equations (2) and (3) are form-invariant under this map.

If we let H be transformed into $\bar{H}$, the requirement that the scalar field equation, and hence the energy density conservation, be satisfied becomes

$$\frac{\bar{x}^2}{x^2} = \left(\frac{H}{\bar{H}}\right)^2 \frac{d\bar{H}}{dH}. \tag{110}$$

In obtaining the latter result the Friedmann constraint equation is also assumed to hold in both frames. Another equivalent condition is also derived from the Raychaudhuri equation

$$\left(\frac{d\bar{\phi}}{d\phi}\right)^2 = \frac{d\bar{H}}{dH}. \tag{111}$$

In addition, from equation (12) we also get

$$\frac{d\bar{\phi}}{\bar{x}\bar{H}} = \frac{d\phi}{xH}, \tag{112}$$

which translates the fact that the form invariance transformation preserves the time variable (14), and which amounts to be the condition that defines the correspondence between the equations of state of the dual models, as characterized by $x(\phi)$ and $\bar{x}(\bar{\phi})$.

We establish a form-invariance transformation between any pair of scalar field solutions by selecting the corresponding generating functions $x(\phi)$ and $\bar{x}(\bar{\phi})$, deriving the corresponding $H(\phi)$ and $\bar{H}(\bar{\phi})$ functions in accordance to equation (12), and plugging them into equation (112) from which we derive the relation $\bar{\phi} = \bar{\phi}(\phi)$. Subsequently, we obtain $\bar{H} = \bar{H}(H)$ by using the conditions (110) and (111).

Notice that by choosing any two functions $x(\phi)$ and $\bar{x}(\bar{\phi})$ we are in fact, due to equation (112), imposing a relation between $\bar{\phi}$ and ϕ. We will show that by considering $x(\phi) = \lambda$ and $\bar{x}(\bar{\phi})$ arbitrary it is possible to give an exact solution for this last case in a very simple way.

A. Proportional Hubble rates

Let us consider the simple case where the Hubble parameters are proportional, that is, $\bar{H} = cH$, where $c \in \mathbb{C}\backslash\{0\}$ is a constant which can be complex. This case can be immediately tackled by resorting to the conditions (110) and (111) to derive

$$\bar{\phi} = \pm\sqrt{c}\phi, \tag{113}$$

and

$$\bar{x} = \pm\frac{1}{\sqrt{c}}x. \tag{114}$$

so that the form-invariance map is

$$\Psi_c : (H, x, \phi) \longrightarrow (cH, \pm\frac{1}{\sqrt{c}}x, \pm\sqrt{c}\phi), \tag{115}$$

which, given equation (17), corresponds to the one given in [35, 36]. Note that from

$$a/a_0 = \exp\left(\int \frac{d\phi}{x(\phi)}\right), \tag{116}$$

one has

$$\bar{a}a^{-c} = \text{const.} \tag{117}$$

The previous general equations permit us to review, in a very simple way, the case analysed by Chimento and Lazkoz from our viewpoint. In accordance to our results of Section III if we choose

the pair $x(\phi) = \lambda$ and $\bar{x} = \bar{\lambda}$, then from equation (112) we have

$$\frac{1}{\bar{H}_0\bar{\lambda}} e^{\frac{\bar{\lambda}}{N-1}\bar{\phi}} d\bar{\phi} = \frac{1}{H_0\lambda} e^{\frac{\lambda}{N-1}\phi} d\phi, \tag{118}$$

from which we derive

$$\bar{\phi} = \frac{\lambda}{\bar{\lambda}}\phi + \bar{\phi}_0 \tag{119}$$

where

$$\phi_0 = \frac{N-1}{\bar{\lambda}}\left[\nu + \ln\left(\frac{\bar{H}_0\bar{\lambda}^2}{H_0\lambda^2}\right)\right] \tag{120}$$

with ν being an arbitrary integration constant. This result subsequently implies that

$$\bar{H} = \left(\frac{\lambda}{\bar{\lambda}}\right)^2 H. \tag{121}$$

We can then distinguish the cases where the constant of proportionality $c = (\frac{\lambda}{\bar{\lambda}})^2$ is equal to ± 1 from those where it takes some other ratio, and we also distinguish the cases where c is positive from those where it is negative. (Naturally the case where $c = -1$ requires that one of the λ parameters be imaginary). Since the time variable is the same in the two solutions which are linked by the form-invariance, the cases where $|c| \neq 1$ correspond to power-law solutions associated with different values of the barotropic-index γ, and hence with different equations of state. The identity transformation which obviously preserves the equation of state corresponds to $c = +1$, i.e., $\bar{x} = \bar{\lambda} = \lambda = x$. The cases where c is negative are quite interesting since they correspond to transformations between standard scalar field barotropic solutions and phantom solutions with a negative kinetic energy. The case when $c = -1$, i.e, $\bar{\lambda} = \pm i\lambda$, was discussed in reference [36], and yields $\bar{a}(t) = a^{-1}(t)$ [46, 47]. When $c < 0$, but $c \neq -1$ we have more general transforms between standard and phantom power-law solutions such $\bar{a}a^{-c} = 1$. The associated phantom form-invariance map

$$\Psi_{-1} : (H, x, \phi) \longrightarrow (-H, -ix, i\phi). \tag{122}$$

gives for any scalar field cosmology its phantom counterpart which is obtained from the Ψ_{-1} map.

At this point it is appropriate to make two remarks. First, if we do not restrict the generating functions x and $\bar{x}$ to be those leading to power-law behaviours from start, the transform $\bar{H} = cH$ does not necessarily constrain the scalar field model to the case of the exponential potential. Second, there is one special application of the proportional Hubble rates transformation this is related to form invariance between $N+1$ and $M+1$ dimensional flat FRW [48]. Indeed, for $c = \frac{\sqrt{6}}{\sqrt{N(N-1)}}$ one can map the solution for $(N+1)$-dimensional space-time to a 3-dimensional one; and for $c = \frac{\sqrt{M(M-1)}}{\sqrt{N(N-1)}}$ from a $(N+1)$-dimensional space-time to a $(M+1)$-dimensional one.

B. Transforming a power-law solution to any other solution

Among all solutions described in Section III the case of the exponential potential, i.e., $x(\phi) = \lambda$ stands out as particularly interesting and simple. Letting $\bar{x}(\bar{\phi})$ arbitrary it is easy to build the Ψ_ϕ map explicitly. Assuming that one chooses the relation between $\bar{x}$ and x to take the form

$$\int_{\bar{\phi}_0}^{\bar{\phi}} \bar{x}(\xi) d\xi = \lambda \phi, \tag{123}$$

it follows that if one invert this relation to obtain $\bar{\phi} = f(\phi)$, where $\tilde{\lambda} = \lambda/(N-1)$, one gets,

$$\bar{\phi} = f(-\tilde{\lambda}^{-1} \ln H/H_0). \tag{124}$$

A simple calculation then yields

$$\frac{d\bar{\phi}}{d\phi} = f'(-\tilde{\lambda}^{-1} \ln H/H_0), \tag{125}$$

and thus

$$\frac{d\bar{H}}{dH} = \left[f'(-\tilde{\lambda}^{-1} \ln H/H_0) \right]^2. \tag{126}$$

Explicitly this map is given by

$$\Psi : (H, x, \phi) \longrightarrow \left(\int dH \left[f'(-\tilde{\lambda}^{-1} \ln H/H_0) \right]^2, \frac{x}{f'(\phi)}, f(\phi) \right). \tag{127}$$

Notice that by using the solution for the $x = \lambda$ case (19) one can construct an asymptotic solution for $\bar{x}$ which can be, in some cases, also a exact solution associated to $\bar{x}$. This is given by

$$\bar{\phi}(t) = f\left(-\frac{N-1}{\lambda} \ln \left(\frac{(N-1)^2}{\lambda^2} (t - t_0) \right) \right). \tag{128}$$

In what follows we consider a couple of examples.

1. Power-law solutions to the intermediate inflationary solutions

In the first case we envisage the possibility of mapping the power-law solutions to the intermediate inflationary solution of Ref. [12], that is the map $x = \lambda$ to $\bar{x} = \lambda/\bar{\phi}$. It is straightforward to obtain the map

$$\Psi : (H, x, \phi) \longrightarrow \left(\frac{\tilde{\lambda}}{\tilde{\lambda} - 2} H^{(1-2/\tilde{\lambda})}, \lambda \exp(-\phi), \exp(\phi) \right). \tag{129}$$

Note that $f(\phi) = \exp(\phi)$.

Then it follows that the exact solution associated to $\bar{x} = \lambda/\bar{\phi}$ is given by, using (128),

$$\bar{\phi}(t) = \left(\frac{N-1}{\lambda^2}\right)^{-(N-1)/\lambda} \cdot (t-t_0)^{-(N-1)/\lambda}, \tag{130}$$

with the potential

$$V(\phi) = A\left(\frac{N(N-1)}{2} - \frac{\lambda^2}{2\phi^2}\right)\phi^{-2\lambda/(N-1)}. \tag{131}$$

2. Power-law solutions to the new exact solution of subsection III

Let us now turn to the case of $x = \lambda$ and $\bar{x} = -(N-1)W'(\bar{\phi})$. In this case it is straight forward to obtain

$$f(\phi) = -\frac{\lambda}{N-1}\phi e^{-\frac{\lambda}{N-1}\phi}, \tag{132}$$

and thus

$$\Psi : (H, x, \phi) \longrightarrow \left(H^{-H\lambda/(N-2)}, \frac{\lambda}{f'(\phi)}, f(\phi)\right). \tag{133}$$

Then it follows that the exact asymptotic solution associated to $\bar{x} = -(N-1)W'(\bar{\phi})$ is given by, using (128),

$$\bar{\phi}(t) = \frac{(N-1)^2(t-t_0)}{\lambda^2}\ln\left(\frac{(N-1)^2(t-t_0)}{\lambda^2}\right), \tag{134}$$

with the potential

$$V(\phi) = A\left(\frac{N(N-1)}{2} - \frac{(N-1)^2W'(\phi)^2}{2}\right)\phi^{2W(\phi)}. \tag{135}$$

Expression (134) extends the solution (89) to the $N \neq 3$ case.

3. A non-homogeneous transformation

In [49] Parsons and Barrow investigated a transformation which also permits to generate an infinite family of solutions for the $k = 0$ FRW scalar field cosmologies for $N = 3$. Theirs is a particular class of form-invariance characterized by $\bar{H} = \alpha^2 H + \beta$, and hence $\bar{\phi} = \alpha\phi$, where both α and β are constants. Applying the equations (111) and (112) of our procedure, we find

$$\bar{x} = x\frac{\alpha H}{\alpha^2 H + \beta}, \tag{136}$$

so that, after choosing $x(\phi)$ which specifies both $H(\phi)$ and $V(\phi)$, given by equations (10) and (12) derive successively $\bar{H}(\bar{\phi})$ and $\bar{x}(\bar{\phi})$. This class of form-invariance transforms illustrates the

implication of adding a constant to the original Hubble rate H, that is of a non-homogeneous form-invariance transformation. In [49] the authors considered a transform relating old and new inflation, and in a subsequent work Barrow, Liddle and Pahud [50] applied the latter invariance to derive a generalization of the intermediate inflation solution.

If we consider the case of $x = \lambda$ and the map (136), which thus reads

$$\bar{x}(\bar{\phi}) = \frac{\lambda}{\alpha} \frac{e^{-\lambda\bar{\phi}/(2\alpha)}}{e^{-\lambda\bar{\phi}/(2\alpha)} + \beta} \tag{137}$$

we get the corresponding potential, recovering the result of Ref. [49],

$$\bar{V}(\bar{\phi}) = \bar{A}\left[3\beta^2 + \alpha^2\left(6\beta - \frac{\lambda^2}{2}\right)e^{-\lambda\bar{\phi}/(2\alpha)} + 3\alpha^4 e^{-\lambda\bar{\phi}/(2\alpha)}\right]. \tag{138}$$

Also for the case $x = \lambda/\phi$ the generating function

$$\bar{x}(\bar{\phi}) = \lambda^2 \frac{\alpha(\bar{\phi}/\alpha)^{-\lambda/2-1}}{\left(\beta + \alpha^2(\bar{\phi}/\alpha)^{-\lambda/2}\right)} \tag{139}$$

yields the potential

$$\bar{V}(\bar{\phi}) = \bar{A}\left[3 - \lambda^2 \frac{\alpha^2(\bar{\phi}/\alpha)^{-\lambda-2}}{\left(\beta + \alpha^2(\bar{\phi}/\alpha)^{-\lambda/2}\right)^2}\right](\bar{\phi}/\alpha)^{-\lambda}, \tag{140}$$

in accordance to Ref. [50].

V. SLOW-ROLL, PERTURBATIONS AND FORM-INVARIANCE MAP

As a joint application of both Sections III and IV we discuss the solutions associated with the potentials that preserve the slow-roll approximation; this can be done explicitly by imposing restrictions on the form-invariance map.

It is known that exponential potentials lead to perturbation spectra that are exact power laws [7]. In [51] a step was given for a systematic classification of types of inflationary potentials that yield a constant scalar perturbation indices. The authors obtain the solutions associated with these potentials for the Harrison-Zel'dovich case and to general power-laws case both to lower order and to next order slow-roll approximation.

It is possible to think of a infinite hierarchy of expressions for the perturbation spectra and for spectral indices. Due to the complexity of the problem, only the first two approximation orders are available in general. To obtain the restriction on the form-invariance map it suffices to consider the first order of approximation in the slow-roll parameters. The scalar and tensor indices are given

by the expressions

$$n_S - 1 \simeq -4\epsilon - 2\eta, \tag{141}$$

$$n_T \simeq= -2\epsilon, \tag{142}$$

So, using equations (25), we get

$$n_S - 1 \simeq -3x^2 + 2x', \tag{143}$$

$$n_T \simeq -x^2. \tag{144}$$

Since the slow-roll approximation is the dynamical regime where

$$\epsilon \ll 1, \tag{145}$$

$$\eta \ll 1, \tag{146}$$

it imposes restrictions on $x = x(\phi)$.

Consider x and $\bar{x}$, where x corresponds to a slow-roll solution, then using the form-invariance map one has

$$\bar{\epsilon} = \frac{x^2}{2f'(\phi)^2}, \tag{147}$$

$$\bar{\eta} = \eta - \frac{x^2}{2}\left(1 + \frac{1}{f'}\right) + x'\left(1 - \frac{1}{f'^2}\right) + x\frac{f''(\phi)}{f'^2}, \tag{148}$$

where f is the function defined by equation (124). This shows that the form-invariance map preserves the slow-roll approximation provided that the following condition is satisfied

$$\left| x'\left(1 - \frac{1}{f'^2}\right) - \frac{x^2}{2}\left(1 + \frac{1}{f'}\right) + x\frac{f''(\phi)}{f'^2} \right| \ll 1. \tag{149}$$

VI. CONCLUSIONS

In this work we have presented a unified mechanism that generates exact solutions of scalar field cosmologies by quadratures. The procedure investigated here permits to recover almost all known exact solutions, and shows how one may derive new solutions. In particular, we have derived one novel solution defined in terms of the Lambert function.

The solutions are organised in a classification which depends on the choice of a generating function which we have denoted by $x(\phi)$. The choice of the latter reflects the underlying thermodynamics of the model. Cases in which $x(\phi)$ differs only by an additive constant correspond to the same potential $V(\phi)$. Conversely, this shows that the selection of a potential does not fixes the

thermodynamical state of the universe. This is a limitation that must be faced by the efforts of reconstructing the potential from observations.

We have also discussed how one can transform solutions from one class, i.e., characterized by a given choice of $x(\phi)$ into solutions belonging to other classes. This type of mappings have been termed form-invariance transformations in the literature [36]. In the present work we have extended these transformations to include all sorts of solutions and space dimensions. In particular we have generalised Chimento and Lazkoz's results on the duality of standard/phantom solutions of power-law models characterized by exponential potentials. We have, for instance, shown how one can transform these power-law solution into intermediate inflationary solutions, or into super-inflationary solutions either phantom or not.

We must emphasise that the possibility of establishing a unified procedure to derive exact scalar field solutions, and to map different classes of solutions one into another through form-invariant transformations ultimately stems from the fact that the field equations are a canonical dissipative system in which the dissipative term is proportional to the square root of the energy of the scalar field as revealed by the generalised Klein-Gordon equation and by the Friedmann constraint equation.

In forthcoming works we extend the procedure and dualities investigated in the present work to exact phantom solutions and to solutions of scalar-tensor gravity theories with a perfect fluid [52].

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